ON NIL SUBRINGS

BY

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ABSTRACT

The nil subrings of rings which satisfy certain ascending chain condition on anihilators are shown to be nilpotent.

The following result has been announced recently [1] :

THEOREM. *Let R be a ring satisfying the ascending chain condition on right and left annihilators, then the nil subrings of R are nilpotent.*

Among the papers of the late Professor Levitzki there are two results from which this theorem follows. The method shows that one can actually require the maximum condition on annihilators of a special type.

NOTATIONS. Let S be a subset of R. Denote by $S_r = \{x; x \in R,\ Sx = 0\}$ the right annihilator of S; similarly, the left annihilator will be denoted by $S_l = \{x; x \in R, xS = 0\}$.

If S is a subring of R, $L(S)$ will denote the Lower Radical ([2]) of S, and $N(S)$ will be the sum of all nilpotent ideals *in* S.

LEMMA 1. *If R satisfies the maximum condition on sequences of annihilators of the form (A): $(b_1)_r \subseteq (b_2)_r \subseteq \ldots$, where $b_{i+1} = b_i r_i b_i$ for arbitrary $r_i \in R$, then the relation $S = L(S)$ holds for every nil subring S of R.*

Proof. If $S \neq L(S)$, let $b \notin L(S)$ and b and element of S. Since $S/L(S)$ does not contain nilpotent ideals, it follows that $(bS)^2 \nsubseteq L(S)$. Hence, there exist $s \in S$ such $bsb \notin L(S)$. Now $sb \in S$ and, therefore, it is nilpotent. Consequently $b(sb)^n = 0$ and $b(sb)^{n-1} \neq 0$ for some integer $n \geqq 2$. In particular this implies that $b_r \subset (bsb)_r$, since $(sb)^{n-1}$ belongs to the second annihilator and does not belong to the first one.

The proof of the lemma follows now readily, by starting with $b = b_1 \notin L(S)$, and $b_2 = b_1 s b_1$ etc., and one is led to an increasing sequence of annihilators $(b_1)_r \subset (b_2)_r \subset (b_3)_r \ldots$ which is a contradiction. Consequently $S = L(S)$.

LEMMA 2. *If R satisfies the increasing chain condition on right and left annihilators of the form:*

$$(A_1)_t \subset (A_1A_2)_t \subset (A_1A_2A_3)_t \ldots$$

where $t = r, l$ and where A_k is a nil ring, and $A_1A_2 \ldots A_k \subseteq A_1A_2 \ldots A_{k-1}$, then for every subring S, the ideal $N = N(S)$ is nilpotent.

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* Professor J. Levitzki died in 1956. This result has been found among his papers and was arranged for publication by S. A. Amitsur.

Proof. First we note for further reference that since $N(S)$ is the union of nilpotent ideals in S, the ideals of S generated by single elements of $N(S)$ are nilpotent, as they are subsets of a finite sum of nilpotent ideals.

Consider now the non-decreasing sequence: $N_l \subseteq N_l^2 \subseteq \dots \subseteq N_l^m \dots$. It follows from the condition of the lemma that $N_l^k = N^{k+1}_l = N^{k+2}_l = \dots$ for some integer k. Put $M = N^k$, then $M_l = M_l^m$ for all integers $m \geqq 1$. Let $P = M \cap M_l$. If N is not nilpotent, then $M \neq 0$ and $P \neq M$, since otherwise $0 = PM = M^2 = N^{2k}$.

Let A be a two sided ideal *in* M such that $A \nsubseteq P$, we assert first that in this case $AM^n \nsubseteq P$ for every $n > 0$. Indeed, if $AM^n \subseteq P$ then $AM^{n+1} \subseteq PM = 0$, and hence $A \subseteq M^{n+1}_l = M_l$, which yields $A \subseteq M_l \cap M = P$ and thus a contradiction. We use this result to show that if N is not nilpotent we obtain an infinite increasing chain of annihilators of the type stated in the lemma:

If N is not nilpotent than $P \neq M$ and we can choose $a_1 \in M$ and $a_1 \notin P$. Thus the ideal $A_1 = (a_1)$ generated by a_1 *in* M satisfies $A_1 \nsubseteq P$ and therefore $A_1 M \nsubseteq P$. Thus, there exist $a_2 \in M$ such that $A_1 a_2 \nsubseteq P$. Let $A_2 = (a_2)$ be the ideal *in* M generated by a_2; then, similarly, $A_1 A_2 \nsubseteq P$ and we obtain an element a_3, such that $A_1 A_2 A_3 \nsubseteq P$. Continuing by this method we obtain a sequence of ideals in $M : A_1, A_2, \dots, A_n, \dots$, each generated by a single element and such that for every $n \geqq 1$, $A_1 A_2 \dots A_n \nsubseteq P$. Now, since $A_1 \dots A_{n-1}$ is an ideal in M, it follows that $A_1 A_2 \dots A_n \subseteq A_1 A_2 \dots A_{n-1}$ and therefore $(A_1 A_2 \dots A_{n-1})_r \subseteq (A_1 \dots A_n)_r$. On the other hand, each A_j is nilpotent, being generated by a single element, hence there exist an integer $m \geqq 2$ such that $A_1 A_2 \dots A_n^m = 0$ and $A_1 A_2 \dots A_n^{m-1} \neq 0$; consequently, $A_n^{m-1} \subseteq (A_1 A_2 \dots A_n)_r$ whereas $A_n^{m-1} \nsubseteq (A_1 \dots A_{n-1})_r$. This leads to the increasing chain

$$(A_1)_r \subset (A_1 A_2)_r \subset \dots \subset (A_1 A_2 \dots A_n)_r \dots$$

which is a contradiction.

The proof of the theorem follows now readily. If the chain condition holds for annihilators then $S = L(S)$ for a nil subring S, and since $N(S)$ is nilpotent it follows that $N(S) = L(S)$ and thus S is nilpotent.

Remark. If a ring R satisfies the chain condition (B) for two sided ideals A_i of R, then the same proof shows a well-known result of the author that for these rings R, $L(R) = N(R)$ is nilpotent.

This follows immediately from the fact that the A_i chosen in the proof of lemma 2 can be chosen to be ideals in R.

Bibliography

1. Herstein, I. N. and Small, L. W., 1963, Nil rings satisfying certain chain conditions, *Notices Amer. Math. Soc.*, **10**, p. 662.

2. Jacobson, N., 1956, Structure of Rings, *Amer. Math. Soc. Colloquium Publ.* No. 37, Ch. VIII, p. 193.

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